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SPIRES AND FAN VAULTS

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Abstract—Shell equations for spires and for fan vaults are of similar form. A brief discussion is made of the masonry spire under dead load, wind load, and vibrating conditions, and membrane solutions are then established for the fan vault. It is shown that such a vault is very stable, and requires less external buttressing than the rib vault.

THE MASONRY SPIRE

RECENT papers [1–3] have applied the limit theorems of plastic design [4, 5], originally formulated for steel framed structures, to masonry construction. It was shown that safe results can be derived on the assumption that masonry can carry compressive stresses, but is incapable of tension (i.e. the weak tensile strength of mortar in masonry joints is ignored). In particular, the membrane theory of shells was adapted for the analysis of domes [3], and it was found that stability was assured if a thrust surface could be found lying wholly within the masonry, but not necessarily coinciding with the centre line of the shell.

The masonry spire is almost invariably octagonal in cross-section, and the corresponding conventional membrane for shell analysis would therefore be a pyramid with eight flat sides. If such a membrane solution (i.e. without bending) is sought, then the spire surface at any point can only carry loads acting in its own plane; in addition, discontinuities will exist at the junctions of the flat surfaces. There are thus severe objections to the use of the pyramid as a model for the analysis of real spires.

If the masonry thickness is not too small compared with the mean "radius" of the spire, then a right circular cone can be contained wholly within the actual shell; if a membrane solution can be established for the cone, then the limit theorems applicable to masonry will ensure that the real structure is at least as safe as the hypothetical cone. The condition that a circle of radius R can be contained within an octagon of wall thickness t is that $R(1-\cos 22\frac{1}{2}°) < t$, or $t/R > 7{\cdot}61$ per cent. Figure 1 shows Banister Fletcher's drawing [6] of St. Andrew, Heckington; the minimum value of t/R, for t about 9 in. in the top half of the spire, is about $10\frac{1}{2}$ per cent.

Figure 2 shows an element of the surface of a right circular cone of half-angle β acted upon by the three membrane stress resultants N_s, N_θ, and $N_{s\theta}$, the loading on the element having been resolved into the three components p_s, p_θ, and p_r (outward normal); the two co-ordinates used to specify the position of the element are s and θ. By resolving in the directions of the load components, the three equations (following Flügge [7]) may be

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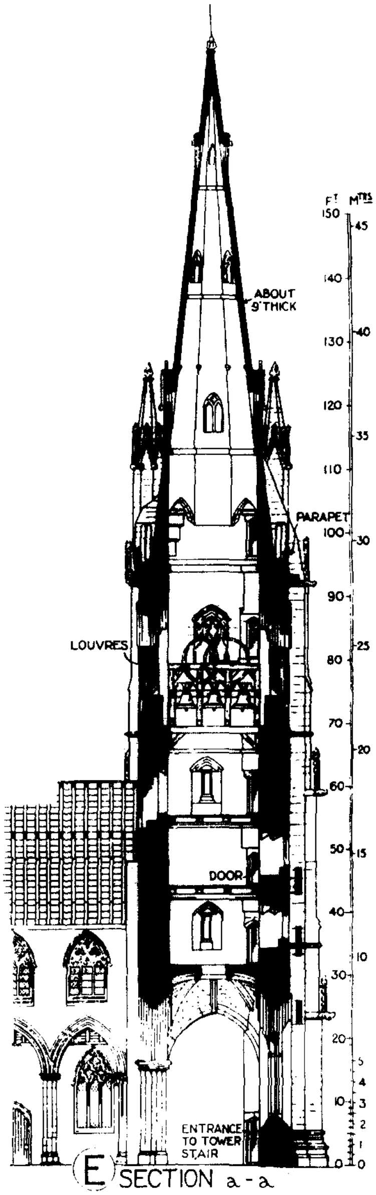

FIG. 1. Sir Banister Fletcher's drawing of St. Andrew, Heckington [6].
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written:

$$\frac{\partial}{\partial s}(sN_s)+\frac{1}{\sin\beta}\frac{\partial N_{s\theta}}{\partial\theta}-N_\theta+sp_s = 0$$

$$\frac{\partial}{\partial s}(sN_{s\theta})+\frac{1}{\sin\beta}\frac{\partial N_\theta}{\partial\theta}+N_{s\theta}+sp_\theta = 0 \quad (1)$$

$$N_\theta = sp_r \tan\beta.$$

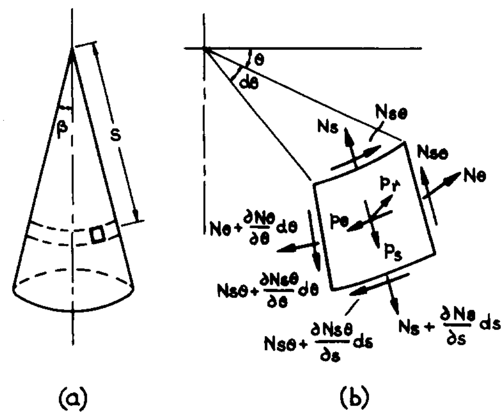


FIG. 2. Conical shell.

Considering first the stresses in the spire due to its own weight, w per unit area, equations (1) may be solved for $p_s = w\cos\beta$, $p_\theta = 0$, $p_r = -w\sin\beta$ to give

$$\begin{aligned} N_s &= -\tfrac{1}{2}ws\sec\beta \\ N_\theta &= -ws\sin^2\beta\sec\beta \\ N_{s\theta} &= 0. \end{aligned} \tag{2}$$

It should be noted that N_θ is given immediately from the third of equations (1), that the second of equations (1) can then be integrated to give $N_{s\theta}$, and finally the first equation can be integrated to give N_s; alternatively, for this symmetrical problem, N_s can be determined without integration by considering the vertical equilibrium of a length s of the spire. For β small, it is clear that N_s is the largest stress resultant, but both N_s and N_θ are always compressive.

It is conventional, and reasonably correct [7], to represent the effect of wind by a normal pressure acting on the spire:

$$\begin{aligned} p_s &= p_\theta = 0 \\ p_r &= -\tfrac{1}{2}p\cos\beta\cos\theta \end{aligned} \tag{3}$$

where p is the usual unit wind pressure specified in standard codes of practice. Introducing these load components into equations (1), it is found that

$$\begin{aligned} N_s &= \tfrac{1}{12}ps\frac{\cos\theta}{\sin\beta}(1-3\sin^2\beta) \\ N_\theta &= -\tfrac{1}{2}ps\cos\theta\sin\beta \\ N_{s\theta} &= -\tfrac{1}{6}ps\sin\theta. \end{aligned} \tag{4}$$

As might be expected, the longitudinal stress resultant N_s is tensile at the windward generator, $\theta = 0$; for β small, N_s is again dominant.

Had the spire been considered as a cantilever, then simple bending theory would have given a maximum bending stress resultant due to wind of magnitude $ps/12\beta$, where the approximation $\sin\beta = \beta$ etc. has been made for a slender spire. The compressive stress resultant due to self weight is $\frac{1}{2}ws$, so that the limiting condition of no tension gives

$$p = 6w\beta \tag{5}$$

independently of the height of the spire. Equation (5) would be derived by simple theory, therefore, as the design wind condition.

As a numerical example, Bond [8] gives details of Louth spire, which is 5 in. thick in its upper part, and for which the half angle β is about 5°. Assuming masonry of unit weight 144 lb/ft^3, then $w = 60$ lb/ft^2, and equation (5) gives $p = (6)(0{\cdot}0873)(60) = 31{\cdot}4$ lb/ft^2 as the largest unit wind pressure that the spire can sustain without tension occurring. Louth spire rises over 200 ft, and the unit wind pressure at exposure D that a modern designer would use is given by CP3 Chapter 5 as 31 lb/ft^2.

A design made according to the condition that tension must not occur has a factor of safety (on wind) of 2; a simple calculation shows that if the condition of equation (5) is just satisfied, then the pressure p must be doubled to cause the spire to overturn about its base. Louth spire appears to be one of the most slender that still survives.

Introducing $p = 6\beta w$ into the shell equations, and approximating $\sin\beta$ by β, then the total stress resultants, obtained by adding equations (2) and (4), are

$$\begin{aligned} N_s &= -\tfrac{1}{2}ws+\tfrac{1}{2}ws\cos\theta \\ N_\theta &= -\beta^2 ws-3\beta^2 ws\cos\theta \\ N_{s\theta} &= -\beta ws\sin\theta. \end{aligned} \tag{6}$$

Numerically, taking a 5-in. shell (Louth) for which $w = 60$ lb/ft^2, and $s = 100$ ft, say

$$\begin{aligned} N_s &= -3000\,(1-\cos\theta) \\ N_\theta &= -6000\,\beta^2\,(1+3\cos\theta) \\ N_{s\theta} &= -6000\,\beta\sin\theta. \end{aligned} \tag{7}$$

Substituting $\beta = 0{\cdot}0873$ (5°), and working directly in stresses (instead of stress resultants), equations (7) become, in lb/in^2 units,

$$\begin{aligned} \sigma_s &= -50\,(1-\cos\theta) \\ \sigma_\theta &= -0{\cdot}76\,(1+3\cos\theta) \\ \sigma_{s\theta} &= -8{\cdot}73\sin\theta. \end{aligned} \tag{8}$$

It is of interest to tabulate these stresses, and also the maximum and minimum stresses, round the spire:

TABLE 1

$\theta°$	σ_s	σ_θ	$\sigma_{s\theta}$	σ_{max}	σ_{min}
0	0	−3·0	0	0	−3·0
30	−6·7	−2·7	−4·4	0·1	−9·5
60	−25·0	−1·9	−7·6	0·4	−27·3
90	−50·0	−0·8	−8·7	0·7	−51.5
120	−75·0	0·4	−7·6	1·1	−75·6
150	−93·3	1·2	−4·4	1·4	−93·5
180	−100·0	1·5	0	1·5	−100·0

It will be seen that weak tensile stresses are developed, due to the small tensile hoop stresses together with the shearing stresses induced by the wind. Although it has been assumed that masonry cannot develop tension, it is certain that the bond in an actual spire, obtained from the mortar and the physical interlocking of the stones, will allow such small tensile stresses to be sustained. Thus, for small β, simple theory will suffice; examination of equations (2) and (4) leads to the conclusion that, if all tension is to be avoided, the value of p is limited to $2\beta w$ instead of $6\beta w$.

VIBRATION OF SPIRES

Oscillations of a spire might be self-excited by the action of a steady wind on the tower and spire, or might be excited by the ringing of bells (particularly in England, where the taste for mathematical sequences rather than melody can lead, in change ringing, to a very regular series of impulses; if the frequency of these impulses is near the natural frequency of the tower plus spire, then large oscillations will result). However the oscillations are induced, they can, if large enough, lead to a fluctuating system of stresses of magnitude sufficient to cause cracking.

For the purpose of analysis, it will be assumed that the spire is moving horizontally in simple harmonic motion of amplitude a and natural (circular) frequency ω_0. Thus the acceleration at every point of the spire is $-a\omega_0^2 \sin \omega_0 t$, and d'Alembert's force on an element of unit area is

$$\frac{wa\omega_0^2}{g} \sin \omega_0 t = f, \quad \text{say.} \tag{9}$$

The load components are therefore

$$\begin{aligned} p_s &= f \cos\theta \sin\beta \\ p_\theta &= -f \sin\theta \\ p_r &= f \cos\theta \cos\beta. \end{aligned} \tag{10}$$

Introducing these values into equations (1), the stress resultants become

$$\begin{aligned} N_s &= -\tfrac{1}{3} fs \cos\theta \operatorname{cosec}\beta \\ N_\theta &= fs \cos\theta \sin\beta \\ N_{s\theta} &= \tfrac{2}{3} fs \sin\theta. \end{aligned} \tag{11}$$

These stress resultants are similar in form and relative magnitude to those due to wind, equations (4); the value of f, given by equation (9), varies with time between equal positive and negative limits.

Using again a "simple" theory for small β, and ignoring the weak tension (similar to that of Table 1), the condition for zero tension at $\theta = 0, \pi$ is that

$$\frac{1}{3}\frac{s}{\beta}|f| < \tfrac{1}{2} ws$$

i.e.

$$a < \tfrac{3}{2}\beta \frac{g}{\omega_0^2}. \tag{12}$$

This result is, of course, independent of the thickness of the spire, but, the greater the half angle β, the greater the amplitude a of oscillation that can be tolerated.

The period of oscillation is unlikely to be greater than 2 sec, nor less than about $\frac{1}{2}$ sec, i.e. ω_0 will lie between about π and 4π. Inequality (12) gives the corresponding maximum amplitudes for Louth ($\beta = 5°$) as 5·1 in. and 0·32 in., with the maximum amplitude for a period of 1 sec as 1·3 in.

Wind stresses and oscillation stresses are additive; church bells should not be rung in a high wind.

FAN VAULTS

Figure 3(a) shows a shell of revolution formed by rotating a curve (of local radius of curvature r_1) about a vertical axis; this shell will later be cut in two by a vertical plane passing through the axis, and the resulting half-shell used as a mathematical model for the fan vault. The two radii of curvature in Fig. 3(a), r_1 and r_2, are related to the co-ordinates (r, φ) by the expressions (see Flügge [7]):

$$\begin{aligned} r_1 \cos\varphi &= -\frac{\mathrm{d}r}{\mathrm{d}\varphi} \\ r_2 \sin\varphi &= r. \end{aligned} \tag{13}$$

The symmetrical condition of self-weight loading will be considered, so that two stress resultants only, N_φ and N_θ, will act, as shown in Fig. 3(b).

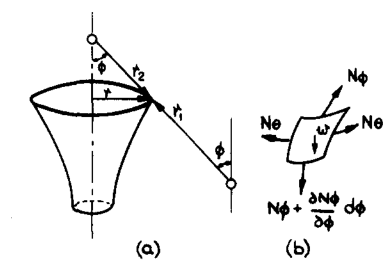


FIG. 3. Complete fan vault conoid.

Had the curve defining the surface of the shell of revolution been a straight line intersecting the axis, the resultant shell would have been a cone standing on its point, i.e. an inverted spire. The actual curvature (r_1) of the surface will lead to some complication, but very similar equations to equations (1) may be written for the fan vault; due to symmetry, the second of equations (1) is satisfied identically, and the remaining two are replaced by

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}(rN_\varphi) + r_1 N_\theta \cos\varphi = -wrr_1 \sin\varphi$$

and

$$\frac{N_\theta}{r_2} = \frac{N_\varphi}{r_1} + w\cos\varphi \tag{14}$$

where, as before, w is the weight per unit area of the shell. The stress resultant N_θ may be eliminated from these equations, and, using equations (13),

$$\frac{d}{d\varphi}(rN_{\varphi}\sin\varphi) = -wrr_1^{\cdot}. \tag{15}$$

Still imagining equation (15) to apply to the complete shell of revolution (and not the half-shell for the fan vault), the solution for a circular profile, Fig. 4, will first be investigated. In Fig. 4(a), the circle just touches the axis, so that the shell comes to a point, and infinite stress resultants must be expected, but the shell solution is in any case fairly meaningless

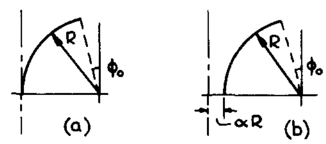


FIG. 4. Fan vault of circular profile.

in this region, as will be seen below. For Fig. 4(a), $r_1 = R =$ const., and equations (13) give $r = R(1-\sin\varphi)$, $r_2 = R(\operatorname{cosec}\varphi - 1)$; equation (15) may be integrated to give

$$N_{\varphi} = -wR\frac{(\varphi+\cos\varphi+C)}{\sin\varphi(1-\sin\varphi)} \tag{16}$$

where C is a constant of integration. The second of equations (14) then gives

$$N_{\theta} = -wR\left[\frac{\varphi+\cos\varphi+C}{\sin^2\varphi} - \frac{\cos\varphi(1-\sin\varphi)}{\sin\varphi}\right]. \tag{17}$$

Examination of expressions (16) and (17) shows that there is no general solution for $\varphi_0 = 0$ in Fig. 4(a), owing to the zeros in the denominators. For the incomplete shell, cut off at φ_0, suppose first that the edge $\varphi = \varphi_0$ is not subject to any external force, i.e. the shell is free-standing. Thus N_{φ} at $\varphi = \varphi_0$ must be zero, and equation (16) gives $C = -\varphi_0 - \cos\varphi_0$. Introducing this value of C into expression (16) and (17), the following table may be constructed giving the stress resultants for $\varphi_0 = 10°$:

TABLE 2
($\alpha = 0$, $\varphi_0 = 10°$, $N_{\varphi} = 0$ at $\varphi = \varphi_0$)

$\varphi°$	10	20	30	40	50	60	70	80	90
N_{φ}/wR	0	−0·57	−0·92	−1·32	−1·98	−3·34	−7·12	−27·5	−∞
N_{θ}/wR	4·68	0·71	−0·05	−0·31	−0·41	−0·44	−0·44	−0·42	−0·41

Very similar results are obtained for the profile of Fig. 4(b), in which the shell is supposed to merge into a column of mean radius αR. Table 3 gives the stress resultants for $\alpha = 0{\cdot}05$:

TABLE 3
($\alpha = 0{\cdot}05$, $\varphi_0 = 10°$, $N_{\varphi} = 0$ at $\varphi = \varphi_0$)

$\varphi°$	10	20	30	40	50	60	70	80	90
N_{φ}/wR	0	−0·57	−0·90	−1·26	−1·79	−2·71	−4·40	−7·34	−9·62
N_{θ}/wR	4·97	0·77	−0·03	−0·31	−0·43	−0·47	−0·48	−0·48	−0·48

In Table 3, the meridional stress resultant N_φ no longer becomes infinite, but the general pattern is the same in both Tables 2 and 3. In particular, while N_φ is always compressive, the hoop stress resultant N_θ is tensile at the upper edge of the shell, a condition which cannot be allowed.

A new solution will therefore be sought for which the hoop stress is everywhere compressive; the boundary condition $N_\theta = 0$ at $\varphi = \varphi_0$ will be used (how such a condition might be achieved in practice is discussed below). Equation (17) for the shell standing on a point then gives $C = \sin\varphi_0 \cos\varphi_0(1-\sin\varphi_0)-\varphi_0-\cos\varphi_0$, and the new stress system is given in Table 4, again for $\varphi_0 = 10°$:

TABLE 4
($\alpha = 0$, $\varphi_0 = 10°$, $N_\theta = 0$ at $\varphi = \varphi_0$)

$\varphi°$	10	20	30	40	50	60	70	80	90
N_φ/wR	−0·99	−1·19	−1·49	−1·94	−2·77	−4·56	−9·61	−37·0	$-\infty$
N_θ/wR	0	−0·50	−0·62	−0·65	−0·65	−0·63	−0·60	−0·57	−0·55

Again, the pattern is not markedly different for the shell supported by a column, Table 5:

TABLE 5
($\alpha = 0{\cdot}05$, $\varphi_0 = 10°$, $N_\theta = 0$ at $\varphi = \varphi_0$)

$\varphi°$	10	20	30	40	50	60	70	80	90
N_φ/wR	−0·99	−1·20	−1·44	−1·83	−2·48	−3·65	−5·85	−9·68	−12·6
N_θ/wR	0	−0·52	−0·64	−0·68	−0·68	−0·67	−0·65	−0·63	−0·63

It is clear that the top of the shell, round to say 30 or 40°, is not "aware" of the support conditions at the base.

The solutions of Tables 4 and 5 now have the required characteristics, in that the stresses are compressive everywhere, but a compressive external load is implied, acting at the "free" edge of the shell, $\varphi = \varphi_0$. Such a compressive load in turn implies that, for a complete fan vault, adjacent fans must "lean" against each other; pendants, heavy keystones, and, in general, the material in the spandrels between the fans, must provide the vertical component of this edge loading.

Suppose now that the complete shell is cut into two equal halves, of which one is shown in Fig. 5. The stress system of Table 4 (or of Table 5) is an equilibrium system for the shell, providing proper external loads are introduced. The half-shell of Fig. 5 must, in fact, be in overall equilibrium; for example, the point load at the base (provided by the supporting column or wall) must equal the weight of the half-shell together with the vertical components of the edge load N_φ at $\varphi = \varphi_0$. These edge loads also have horizontal components, which must be balanced by external horizontal forces acting on the vertical cut edges, supplied by the walls (and, presumably, transmitted to external buttresses; see below). These horizontal forces are, of course, the integrated values of the hoop stress resultant N_θ; if these hoop forces sum to a value F at the cut, then

$$F = 2\int_{\varphi_0}^{\pi/2} N_\theta R \, d\varphi \qquad (18)$$

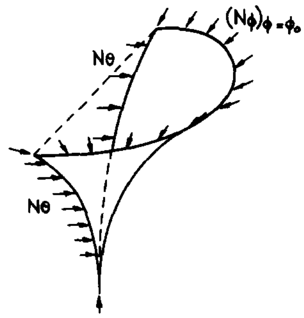


FIG. 5. Overall equilibrium of fan vault.

Thus, if a resultant force of this magnitude is applied (at the proper height above the springing), together with the vertical propping force at the springing, the half-shell will be in equilibrium under an internal state of stress given by Table 4, which was computed for the complete shell. Table 4 does not give the stresses which will necessarily *actually* occur in the half-shell, but it has now been demonstrated that these stresses form a possible equilibrium system for the half-shell. The "safe" limit theorem of structural analysis, applied to masonry [1], states that if any one equilibrium state can be found for which the structure is stable, then this is complete assurance of stability without the need for calculation of the actual stress state. Despite the fact, therefore, that the half-shell of Fig. 5 is no longer a complete shell of revolution, no further analysis need be undertaken.

Substituting the value of N_θ from equation (17) into equation (18), it is found that $F = 2wR^2 \cot\varphi_0[\varphi_0+\cos\varphi_0+C]$, and, using the appropriate value of C, $F = 2wR^2 \cos^2\varphi_0(1-\sin\varphi_0)$. For $\varphi_0 = 10°$, $F = 1{\cdot}60wR^2$, and this value is shown in Fig. 6(a), together with the other forces necessary for equilibrium of the fan. Numerically, if $w = 60\,\text{lb/ft}^2$, $R = 25\,\text{ft}$, the forces are as shown in Fig. 6(b). A dead weight (keystones and the like) of 7 ton is required for each fan; the horizontal thrust of 27 ton is high. The stresses, except near the springing of the fan, are low; if the numbers in Table 4 are multiplied by 25 (for $R = 25$ ft and a material density of $144\,\text{lb/ft}^3$) the resultant figures will give stresses in lb/in^2.

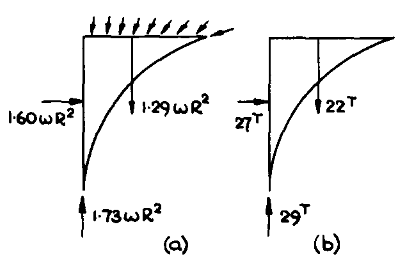


FIG. 6. External forces on fan vault.

All these results have been obtained for an assumed circular profile which is, in one sense, not a very good approximation to the shape of actual fan vaults. In another sense, a *membrane* (of vanishingly small thickness) of circular profile can be contained within the real thickness of a wide range of actual vaults. In the present context, the problem of shell analysis is not the determination of stress resultants for the centre surface of the actual vault, but the establishment of a stress system for any membrane that can be contained within the thickness of the vault.

As was seen from Fig. 6, the circular profile led to a high horizontal thrust necessary for equilibrium of the vault. Shell solutions are often extremely sensitive to the assumed profile, and large stress variations can result from very small changes of shape [3]. Such variations can affect not only the values of individual stress components, but can alter radically the whole distribution of stress in a shell. It is possible, therefore, that a "better" distribution (involving, from the practical point of view, smaller thrusts) might be found by investigating other profiles.

As an example, an inverse solution of the shell equations will be sought for which the hoop stress resultant N_θ is everywhere zero. Such a shell could be cut into two fan vaults by a vertical plane (c.f. Fig. 5) without the necessity of any compensating force at the wall; the horizontal component of any compressive load applied to the top edge of the fan must thus be balanced by a thrust at the springing, as will be seen below. (A recent paper by Anderson *et al.* [9] discusses the problem of the best shape for a space vehicle entering thin planetary atmospheres. Effectively, a thin membrane is designed as a sort of parachute, and the governing design requirement is that, to avoid buckling, the stresses everywhere must be *tensile*. This problem is an almost exact analogue of that of the masonry vault, and, although the loading conditions are different, the parachute proposed is remarkably like a complete fan vault, entering the atmosphere point first. At the widest part of the fan, the parachute edge must be reinforced by a stiffening ring; no solution exists for a free edge. This, again, is analogous to the edge compressive load required above for the vault of circular profile; a similar edge load is necessary for the inverse design of vault).

Setting $N_\theta = 0$ in equations (14), and eliminating N_φ, a double integration leads to the intrinsic equation for the shape of the fan:

$$\tan\varphi = \sinh\left(\log_e C - \frac{r^2}{x_0^2}\right) \tag{19}$$

where C and x_0 are constants of integration; the value of N_φ is given by

$$N_\varphi = -\frac{wx_0^2}{2r\cos\varphi} \tag{20}$$

Transferring to the rectangular co-ordinates of Fig. 7, where the origin is taken on the axis of the fan, the constant x_0 may be identified with the maximum radius of the fan,

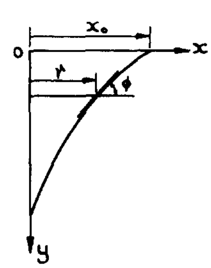


FIG. 7. Axes for fan vault.

for which $y = 0$. From Fig. 7, $\tan\varphi = -\mathrm{d}y/\mathrm{d}x$, and equation (19) may be integrated to give, finally,

$$y = C\frac{x_0}{2}\left[\int_{x/x_0}^{1} \mathrm{e}^{-t^2}\,\mathrm{d}t - \frac{1}{C^2}\int_{x/x_0}^{1} \mathrm{e}^{t^2}\,\mathrm{d}t\right]$$

The shape of the fan determined by equation (21) consists of a single family of curves depending on the value of C; the constant x_0 is merely a scale factor (which might be thought of, for the moment, as the half-width of the nave). The first integral in the equation (21) is essentially the error integral, tabulated by Jahnke and Emde [10], who also give a brief table of the second integral. From equation (19), φ is positive at $r = x_0$ if $C > \mathrm{e}$, and this is the lower bound for C of practical interest. In Fig. 8 are drawn profiles of fans for $C = \mathrm{e}$, 3, 3·24($\varphi_0 = 10°$), and 4, and superimposed is the approximate outline of the fan vault of King's College chapel (half-nave width).

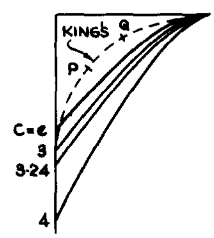


FIG. 8. Inverse profiles.

Three of the profiles are further displayed in Fig. 9, where the overall equilibrium conditions are shown (all values should be multiplied by wx_0^2). The horizontal thrust exerted by the vault has value wx_0^2, independent of the value of C; comparing with Fig. 6 it will be seen that the thrust acts much lower than in the fan of circular profile. Mackenzie [11] gives the thickness of King's vault as varying between 2 and 6 in. Taking 5 in. ($w = 60$ lb/ft^2 as before) and $x_0 = 20$ ft, $wx_0^2 = 10{\cdot}7$ ton, which is a reasonably low value

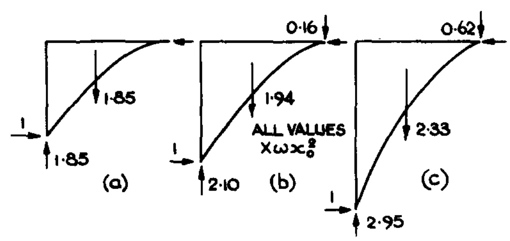


FIG. 9. External forces on inverse profile vaults.

of thrust. From equation (20), the meridional stress resultant N_φ, always compressive, increases in value from about $\frac{1}{2}wx_0$ at the top of the fan to infinity at the springing; again, a real fan will be merged into a supporting wall or column. For $x_0 = 20$ ft, therefore, the edge compressive stress, applied externally by the weight of the spandrel, will be about 10 lb/in^2.

Howard [12] classifies fan vaults into those with separate conoids, Fig. 10, in which case the spandrel may take several alternative forms, Fig. 11, and those with intersecting conoids, Fig. 12. In fact, Fig. 12 merely shows a further alternative for the spandrel, whose

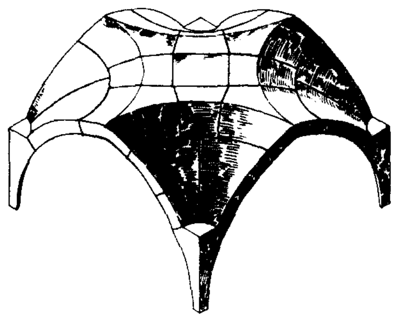


FIG. 10. Fan vault with separate conoids (F. E. Howard).

surface happens to be continuous with the surfaces of the conoids. All cases may be reduced to the plan of Fig. 13, in which the fans, of semicircular plan, carry masonry spandrels, shown shaded. Considering the fan and spandrel to be two separate structural elements, the relatively flat spandrel, approaching as it does to the plate-bande, is extremely stable [1]. Further, the thrust surface for the spandrel will adjust itself without difficulty to the edge thrusts required by the adjacent fans, since a wide range of thrusts will produce only small changes in thrust surface for such small rise structural elements [3].

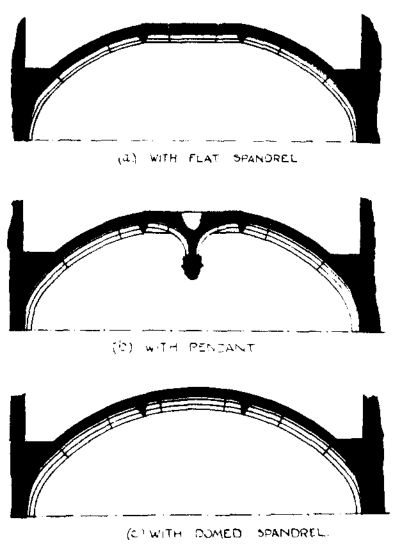
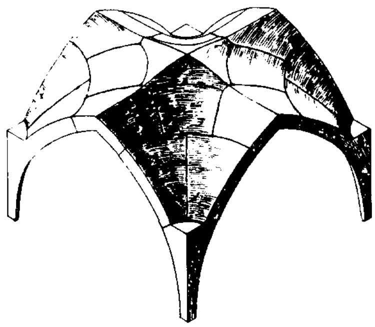


FIG. 11. Alternative spandrels (F. E. Howard).

Figures 10, 12 and 13 have tacitly assumed square vaulting bays (which best suit the fan vault), but most naves do not have square compartments. Bond's [8] drawing of Sherborne nave, Fig. 14, shows a typical rectangular arrangement, and a division into fan plus spandrel would result in the plan shown in Fig. 15. This arrangement is followed,



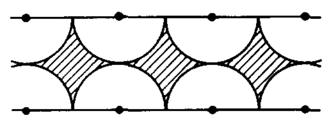
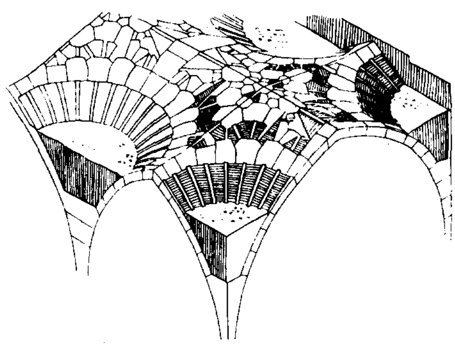


FIG. 12. Fan vault with intersecting conoids (F. E. Howard). FIG. 13. Fan-vaulted nave; square compartments

in effect, in King's College Chapel (although this is obscured by the heavy and useless transverse arches marking each vaulting bay); Henry the Seventh's Chapel at Westminster introduces pendants away from the walls, from which spring the fans, converting rectangular into almost square compartments.



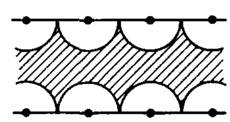


FIG. 14. Francis Bond's drawing of Sherborne nave. FIG. 15. Fan-vaulted nave; rectangular compartments.

In Fig. 14 may be seen the rubble fill at Sherborne in the vaulting conoid, extending about half-way up; similar fills are present in King's College Chapel, St. George's Windsor, and, indeed, in the vaulting conoids formed by the ribs of many Gothic rib vaults (e.g. Beauvais, Amiens), and have been commented upon by Fitchen [13]. Such a fill enables the thrust surface to pass out of the fan vault shell into the solid interior. Thus the "true" fan at King's College Chapel is not at all as shown by the full profile in Fig. 8, which is merely the profile of half the nave, but is more nearly only the segment PQ of that profile. It is evident that such a short segment, having the real thickness of the vault, can easily contain a range of the inversely derived profiles corresponding to a range of values of the constant C. That is, it will certainly be possible to find, for the portion PQ of the fan, a membrane, lying wholly within the masonry, for which the hoop stress resultant is zero.

Finally then, Fig. 16 shows a rectangular vaulting compartment divided into the three regions for structural analysis. On the centre line, the spandrel of shallow or zero rise spans between the fans, and carries bosses and pendants at the whim of the designer; as mentioned above, the spandrel will be very stable, even for thin masonry shells. The fans proper are of limited extent, and a profile of any "reasonable" shape and quite small thickness can accommodate the curve of equation (21). At the wall, the fill to the vaulting conoid allows the vault thrust to act within a range of levels.

However, there will always *be* a horizontal thrust, just as for the essential Gothic structure of the rib vault kept in equilibrium by the flying buttress; the fan vault also requires external buttressing to ensure overall equilibrium. The horizontal thrust was noted in Fig. 9, and its line of action will be dropped further due to the presence of the rubble fill in the conoid; indeed, the line of action may drop far enough for flying buttresses to be omitted altogether, as at Sherborne nave (but not Sherborne choir). At King's, the enormous external main buttresses absorb the vault thrust.

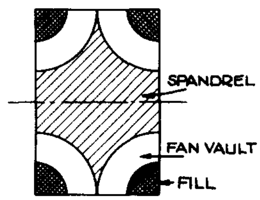


FIG. 16. Rectangular vaulting compartment.

If, then, a fan vault is built without flying buttresses, the thrust surface must drop to a level where the horizontal thrust can emerge from the vault into whatever buttressing is in fact provided. The continuous turning surface of the fan, together with the solid lower portion of the conoid, enables the fan to construct for itself such a lowered thrust surface. By contrast, the thrust of a Gothic rib vault is collected, in the main, into the diagonal ribs [1, 2], and very little adjustment can be made to the line of action of the resultant total horizontal thrust; the flying buttress must be placed in approximately the optimum position, or the rib vault will collapse.

Effectively, then, a late fifteenth-century masonry designer specializing in fan vaults (King's was vaulted in 1512–15 by John Wastell) could pick any likely-looking profile, could decorate the surface of the vault with non-structural ribs, could insert on the centre line of the nave a series of heavy bosses or pendants and could pay relatively scant attention to the external buttressing system, all with almost complete assurance that his structure would be stable. Indeed, the continued existence of such vaults, despite settlements and other distortions of the whole fabric, is ample experimental evidence that the masonry vault is an essentially stable structure.

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Résumé—Les équations du genre "membrane" pour les flèches et pour les voûtes en éventail se ressemblent. Une discussion brève est donnée au sujet des effets sur la flèche en maçonnerie de la pesanteur, du vent, et des conditions oscillatoires: ensuite des solutions sont établies pour la voûte en éventail. On démontre qu'une telle voûte est très solide et exige moins d'arc-boutement que la voûte à nervures.

Zusammenfassung—Schalengleichungen für Türme und Fachgewölbe haben ähnliche Formen. Mauerwerk-Türme unter stehender Last, Windlast und unter Vibration werden kurz behandelt; ferner werden Lösungen für das Fachgewölbe gegeben. Es wird gezeigt, daß ein derartiges Gewölbe sehr stabil ist und weniger Strebepfeiler braucht als das Rippengewölbe.

Абстракт—Уравнения оболочки для шпилей и для веерных сводов—одинаковой формы. Кратко обсуждается шпиль каменной кладки под мёртвой нагрузкой, ветровой нагрузкой и под условиями вибрации и затем установлены решения мембраны для веерного свода. Показано, что такой свод очень устойчив и требует меньше внешней контрфорсной подпорки, чем ребристый свод.